1. Let $r \in (0,1)$. Show that

(i) Let $\delta > 0$ be such that $1+\delta = \frac{1}{r}$. Show that $(0<) r^n \leq \frac{1}{1+n\delta}$ (Hint: Binomial or Bernoulli).

(ii) Show that $\lim_n r^n = 0$ (Hint: Squeeze).

2. Let $r \in (0,1)$ and

$$s_n := 1 + r + r^2 + \cdots + r^n \quad (n \in \mathbb{N}).$$

Show that $s_n = \frac{1-r^{n+1}}{1-r}$ $\forall n \in \mathbb{N}$ and

$$\lim_n s_n = \frac{1}{1-r}$$

3. Let $c \in (0,1)$, and let (x_n) be a c-contraction sequence, namely

$$|x_{n+1} - x_n| \leq c\,|x_n - x_{n-1}|, \quad \forall n \in \mathbb{N} \setminus \{1\}.$$

Show, by MI, that, $\forall n \in \mathbb{N}$

(i) $|x_{n+1} - x_n| \leq c^{n-1} |x_2 - x_1|$

(convention : $c^n = 1$ if $n = 0$),

and so

(ii) $|x_{n+j} - x_{n+j-1}| \leq c^{n+j-2} |x_2 - x_1|, \; \forall j \in \mathbb{N}$

and

(iii)
$$|x_{n+j} - x_n| \leq \left(c^{n+j-2} + c^{n+j-3} + \cdots + c^{n-1}\right) |x_2 - x_1| \leq \frac{c^{n-1}}{1-c} |x_2 - x_1|$$

Consequently, show further that (x_n) is a Cauchy sequence.

4*. Let
$$x_{n+1} = 2 + \frac{x_n}{2} \quad \forall n \in \mathbb{N} \cancel{\setminus \{1\}}.$$

Then, for each of the following cases, show that (x_n) converges (and find the value of the limit):

(i) $x_1 = 0$;

(ii) $x_1 = 10$.

(Hint: Can the MCT be applied?)

5*. Show that $\lim\limits_{n} \dfrac{n^7}{(1+\delta)^n} = 0$ (where $\delta > 0$).

Hint (similar to Q1 but expand more terms when apply the Binomial).

6*. Let $x_1 > 0$ and

$$x_{n+1} = x_n + \frac{1}{x_1} \quad \forall \mathbb{N} \cancel{\setminus \{1\}}.$$

Use two methods below to show that (x_n) does <u>not</u> converge:

(a) Use Q6 of HW 2.

(b) Use (algebraic computation rules).

7*. Suppose $\lim\limits_{n} y_n = y$. Show

(i) If $y > 0$ then there exists $N \in \mathbb{N}$ such that

$$0.9 \cdot y < y_n < 2y \quad \forall n \geq N.$$

$\left(\frac{9}{10} y < y_n \cdots\right)$.

(ii) If $y \neq 0$ then there exists $N \in \mathbb{N}$ such that

$$0.9 \cdot |y| < |y_n| < 2|y|, \forall n \geq N.$$

(iii) Suppose $\lim_n y_n = y$, $y \neq 0$ and $\delta \in (0, |y|)$. Then $\exists\, N \in \mathbb{N}$ s.t.

$$(1-\delta)|y| < |y_n| < \frac{1}{3} + |y| \quad \forall n \geq N.$$